Transition from anticipatory to lag synchronization via complete synchronization in time-delay systems

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The existence of anticipatory, complete, and lag synchronization in a single system having two different time delays, that is, feedback delay τ_1 and coupling delay τ_2, is identified. The transition from anticipatory to complete synchronization and from complete to lag synchronization as a function of coupling delay τ_2 with a suitable stability condition is discussed. In particular, it is shown that the stability condition is independent of the delay times τ_1 and τ_2. Consequently, for a fixed set of parameters, all the three types of synchronizations can be realized. Further, the emergence of exact anticipatory, complete, or lag synchronization from the desynchronized state via approximate synchronization, when one of the system parameters (b_2) is varied, is characterized by a minimum of the similarity function and the transition from on-off intermittency via periodic structure in the laminar phase distribution.

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I. INTRODUCTION

Synchronization phenomena date back to the period of Huygens in 1665, when he found that two very weakly coupled pendulum clocks, hanging from the same beam, become phase synchronized [1]. Since the early identification of synchronization in chaotic oscillators [2–4], the phenomenon has attracted considerable research activity in different areas of science [1,5,6] and several generalizations and interesting applications have been developed. The chaos synchronization phenomenon is of interest not only from a theoretical point of view but also has potential applications in diverse subjects such as as biological, neurological, laser, chemical, electrical, and fluid mechanical systems as well as in secure communication, cryptography, and so on [1–10]. A recent review on the phenomenon of chaos synchronization can be found in Ref. [11]. In recent years, different kinds of synchronization have been identified: Complete (or identical) synchronization [2,3], generalized synchronization [12–14], phase synchronization [15,16], lag synchronization [17–19], and anticipatory synchronization [20–22]. For a critical discussion of the interrelationship between various kinds of synchronization, we may refer to Refs. [23,24]. Transition from one kind of synchronization to another, the coexistence of different kinds of synchronization in time series, and also the nature of the transition have also been studied extensively [17–19,25,26] in coupled chaotic systems.

One of the most important applications of chaos synchronization is secure communication. It is now an accepted fact that secure communication based on simple low-dimensional chaotic systems does not ensure a sufficient level of security, as the associated chaotic attractors can be reconstructed with some effort and the hidden message can be retrieved by an eavesdropper [27]. One way to overcome this problem is to consider chaos synchronization in high-dimensional systems having multiple positive Lyapunov exponents. This increases security by giving rise to much more complex time series, which are apparently not vulnerable to the unmasking procedures generally. Recently chaotic time-delay systems have been suggested as good candidates for secure communication [28,29], as the time-delay systems are essentially infinite dimensional in nature and are described by delay differential equations, and they can admit hyperchaotic attractors with a large number of positive Lyapunov exponents for suitable nonlinearity. Therefore the study of chaos synchronization in time-delay systems is of considerable practical significance. However, it should be noted that one has to be cautious due to the fact that even in time-delay systems with multiple positive Lyapunov exponents unmasking may be possible. In particular, this is so if any reconstruction of the dynamics of the system is achieved in some appropriate space even for very high-dimensional dynamics as demonstrated by Zhou and Lai [30] in the case of the Mackey-Glass equation.

Time delay is ubiquitous in many physical systems due to the finite switching speed of amplifiers, finite signal propagation time in biological networks, finite chemical reaction times, memory effects, and so on [28,29,31,32]. In recent times, considerable work has been carried out on the effect of time delay in limit cycle oscillators [33,34], time-delay feedback [35,36], networks with time-delay coupling [37], etc. Recently, we have shown that even a single scalar delay equation with piecewise-linear function can exhibit hyperchaotic behavior even for small values of the time delay [38]. It is therefore of importance to consider the synchronization of chaos in such scalar piecewise-linear delay differential systems with appropriate delay coupling. Interestingly, in the present work we find that in such a coupled system, one can identify anticipatory, complete, and lag synchronizations by simply tuning the second time-delay parameter in the coupling, for a fixed set of system parameters satisfying appropriate stability condition. The results have been corroborated by the nature of similarity functions, and transition behavior characterized by the probability distribution of the laminar phase during approximate synchronization, which precedes

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exact synchronization when a system parameter is varied. We also wish to point out that to our knowledge such transitions between all three types of the above synchronizations have not been reported in nonhyperchaotic systems and it appears that the present type of hyperchaotic systems are convenient tools to realize such transitions by tuning the delay parameters suitably.

Specifically, in this paper we will consider chaos synchronization of two single scalar piecewise-linear time-delay systems studied in Refs. [31,38,39] with unidirectional coupling between them and having two different time delays: one in the coupling term and the other in the individual systems, namely, feedback delay. We have identified the stability condition for synchronization following Krasovskii-Lyapunov theory and demonstrate that there exists a transition between three different kinds of synchronization, namely, anticipatory, complete, and lag synchronizations, as a function of time delay in the coupling. To characterize the existence of anticipatory and lag synchronizations, we have plotted the similarity function $S(\tau)$. We have also demonstrated that when the system parameter b_2 is varied, the onset of exact anticipatory, complete, and lag synchronization from the desynchronized state is preceded by a region of approximate synchronized state. We also show that the latter is characterized by a transition from on-off intermittency to a periodic structure in the laminar phase distribution, as suggested in the work of Zhan *et al.* [19] for the case of lag synchronization. The plan of the paper is as follows. In Sec. II, we introduce the unidirectionally coupled scalar time-delay system and identify the condition for stability of synchronized states. In Sec. III, we point out the existence of anticipatory synchronization when the strength of the coupling delay is less than feedback delay, while in Sec. IV, complete synchronization is realized when the two delays are equal. Lag synchronization is shown to set in when the coupling delay exceeds the feedback delay in Sec. V. Finally in Sec. VI, we summarize our results.

II. PIECEWISE LINEAR TIME-DELAY SYSTEM AND STABILITY CONDITION FOR CHAOS SYNCHRONIZATION

At first, we will introduce the single scalar time-delay system with piecewise linearity and bring out its hyperchaotic nature for suitable values of the system parameters. Then the unidirectional delay coupling is introduced between two scalar systems and the stability condition for chaos synchronization is derived.

A. The scalar delay system

We consider the following first-order delay differential equation introduced by Lu and He [39] and discussed in detail by Thangavel *et al.* [31]:

$$\dot{x}(t) = -ax(t) + bf(x(t-\tau)), \tag{1}$$

where a and b are parameters, τ is the time delay, and f is an odd piecewise-linear function defined as

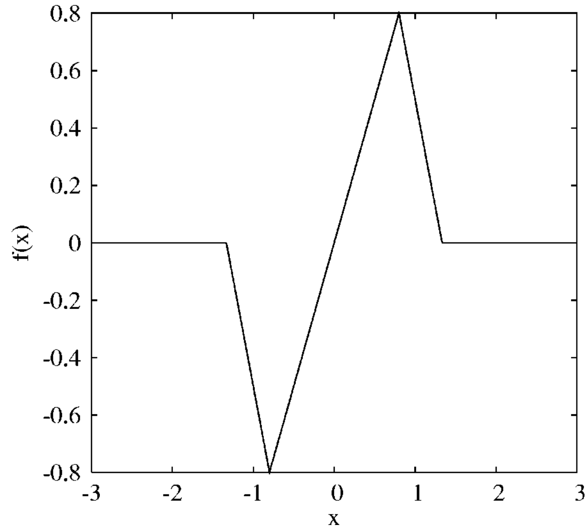

FIG. 1. The schematic form of the piecewise-linear function $f(x)$ given by Eq. (2).

$$f(x) = \begin{cases} 0, & x \leqslant -4/3, \\ -1.5x-2, & -4/3 < x \leqslant -0.8, \\ x, & -0.8 < x \leqslant 0.8, \\ -1.5x+2, & 0.8 < x \leqslant 4/3, \\ 0, & x > 4/3. \end{cases} \tag{2}$$

It is also of interest to consider additional forcing on the right hand side of Eq. (1); however, this is not considered here. The schematic form of (2) is shown in Fig. 1. Recently, we have reported [38] that systems of the form (1) exhibit hyperchaotic behavior for suitable parametric values. For our present study, we find that for the choice of the parameters $a=1.0$, $b=1.2$, and $\tau=25.0$ with the initial condition $x(t)=0.9$, $t\in(-5,0)$, Eq. (1) exhibits hyperchaos. The corresponding pseudoattractor is shown in Fig. 2. The hyperchaotic nature of Eq. (1) is confirmed by the existence of multiple positive Lyapunov exponents. The first ten maximal Lyapunov exponents for the parameters $a=1.0$, $b=1.2$, $x(t)$

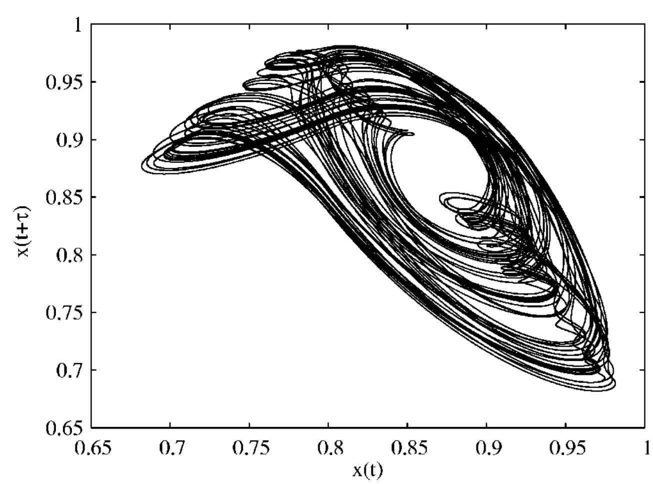

FIG. 2. The hyperchaotic attractor of the system (1) for the parameter values $a=1.0$, $b=1.2$, and $\tau=25.0$.

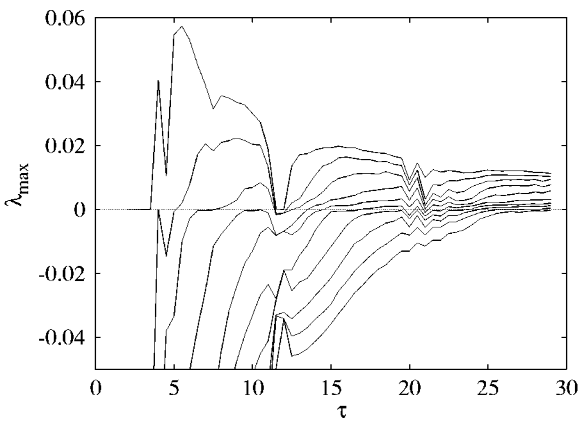

FIG. 3. The first ten maximal Lyapunov exponents λ_{max} of the scalar time-delay equation (3a) for the parameter values $a=1.0$, $b=1.2$, $\tau \in (2,29)$.

$=0.9$, $t \in (-5,0)$ as a function of time delay τ is shown in Fig. 3, which are evaluated using the procedure suggested by Farmer [29].

B. Coupled system and the stability condition

Now let us consider the following unidirectionally coupled drive $x_1(t)$ and response $x_2(t)$ systems with two different time delays τ_1 and τ_2 as feedback and coupling time delays, respectively,

$$\dot{x}_1(t) = -ax_1(t) + b_1 f(x_1(t-\tau_1)), \tag{3a}$$

$$\dot{x}_2(t) = -ax_2(t) + b_2 f(x_2(t-\tau_1)) + b_3 f(x_1(t-\tau_2)), \tag{3b}$$

where b_1, b_2, and b_3 are constants, $a>0$, and $f(x)$ is of the same form as in Eq. (2).

Now we can deduce the stability condition for synchronization of the two time-delay systems Eqs. (3a) and (3b) in the presence of the delay coupling $b_3 f(x_1(t-\tau_2))$. The time evolution of the difference system with the state variable $\Delta = x_{1\tau_2-\tau_1} - x_2$ (which corresponds to anticipatory synchronization when $\tau_2<\tau_1$, identical synchronization for $\tau_2=\tau_1$, and lag synchronization when $\tau_2>\tau_1$), where $x_{1\tau_2-\tau_1}=x_1(t-(\tau_2-\tau_1))$, can be written for small values of Δ by using the evolution equations (3) as

$$\dot{\Delta} = -a\Delta + (b_2+b_3-b_1)f(x_1(t-\tau_2)) + b_2 f'(x_1(t-\tau_2))\Delta_{\tau_1}, \quad \Delta_\tau = \Delta(t-\tau). \tag{4}$$

In order to study the stability of the synchronization manifold, we choose the parametric condition

$$b_1 = b_2 + b_3, \tag{5}$$

so that the evolution equation for the difference system Δ becomes

$$\dot{\Delta} = -a\Delta + b_2 f'(x_1(t-\tau_2))\Delta_{\tau_1}. \tag{6}$$

The synchronization manifold is locally attracting if the origin of this equation is stable. Following the Krasovskii-Lyapunov functional approach [40,41], we define a positive definite Lyapunov functional of the form

$$V(t) = \frac{1}{2}\Delta^2 + \mu\int_{-\tau_1}^{0}\Delta^2(t+\theta)d\theta, \tag{7}$$

where μ is an arbitrary positive parameter, $\mu>0$. Note that $V(t)$ approaches zero as $\Delta\to 0$.

To estimate a sufficient condition for the stability of the solution $\Delta=0$, we require the derivative of the functional $V(t)$ along the trajectory of Eq. (6),

$$\frac{dV}{dt} = -a\Delta^2 + b_2 f'(x_1(t-\tau_2))\Delta\Delta_{\tau_1} + \mu\Delta^2 - \mu\Delta_{\tau_1}^2, \tag{8}$$

to be negative. The above equation can be rewritten as

$$\frac{dV}{dt} = -\mu\Delta^2\Gamma(X,\mu), \tag{9}$$

where $X=\Delta_{\tau_1}/\Delta$, $\Gamma=\{[(a-\mu)/\mu]-[b_2 f'(x_1(t-\tau_2))/\mu]X+X^2\}$. In order to show that $dV/dt<0$ for all Δ and Δ_τ and so for all X, it is sufficient to show that $\Gamma_{min}>0$. One can easily check that the absolute minimum of Γ occurs at $X=(1/2\mu)b_2 f'(x_1(t-\tau_2))$ with $\Gamma_{min}=[4\mu(a-\mu)-b_2^2 f'(x_1(t-\tau_2))^2]/4\mu^2$. Consequently, we have the condition for stability as

$$a > \frac{b_2^2}{4\mu}f'(x_1(t-\tau_2))^2 + \mu = \Phi(\mu). \tag{10}$$

Again $\Phi(\mu)$ as a function of μ for a given $f'(x)$ has an absolute minimum at $\mu=[|b_2 f'(x_1(t-\tau_2))|]/2$ with $\Phi_{min}=|b_2 f'(x_1(t-\tau_2))|$. Since $\Phi \geqslant \Phi_{min}=|b_2 f'(x_1(t-\tau_2))|$, from the inequality (10), it turns out that the sufficient condition for asymptotic stability is

$$a > |b_2 f'(x_1(t-\tau_2))| \tag{11}$$

along with the condition (5) on the parameters b_1, b_2, and b_3.

Now from the form of the piecewise linear function $f(x)$ given by Eq. (2), we have

$$|f'(x_1(t-\tau_2))| = \begin{cases} 1.5, & 0.8 \leqslant |x_1| \leqslant \frac{4}{3}, \\ 1.0, & |x_1| < 0.8. \end{cases} \tag{12}$$

Note that the region $|x_1|>4/3$ is outside the dynamics of the present system [see Eq. (2)]. Consequently the stability condition (11) becomes $a>1.5|b_2|>|b_2|$ along with the parametric restriction $b_1=b_2+b_3$.

Thus one can take $a>|b_2|$ as a less stringent condition for (11) to be valid, while

$$a > 1.5|b_2|, \tag{13}$$

as the most general condition specified by (11) for asymptotic stability of the synchronized state $\Delta=0$. The condition (13) indeed corresponds to the stability condition for exact anticipatory, identical, as well as lag synchronizations for suitable values of the coupling delay τ_2. It may also be noted that the stability condition (13) is independent of the both the delay parameters τ_1 and τ_2. In the following, we will

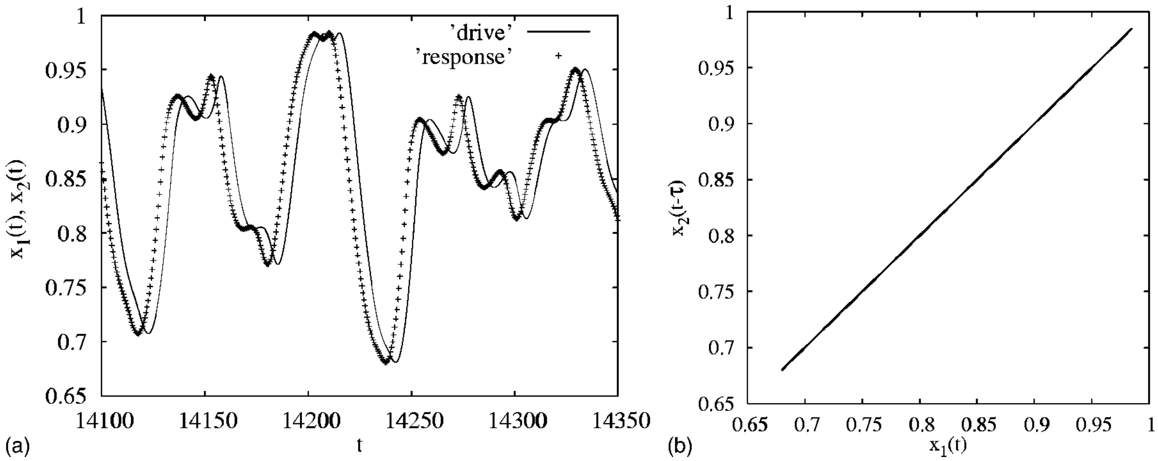

FIG. 4. Exact anticipatory synchronization for the parameter values $a=0.16$, $b_1=0.2$, $b_2=0.1$, $b_3=0.1$, $\tau_1=25.0$, and $\tau_2=20.0$. (a) Time series plot of $x_1(t)$ and $x_2(t)$; (b) synchronization manifold between $x_1(t)$ and $x_2(t-\tau)$, $\tau=\tau_2-\tau_1$. The response $x_2(t)$ anticipates the drive $x_1(t)$ with a time shift of $\tau=5.0$.

demonstrate the transition from anticipatory to lag synchronization via complete synchronization as the coupling delay τ_2 is varied from $\tau_2<\tau_1$ to $\tau_2>\tau_1$, subject to the stability condition (13) with the parametric restriction $b_1=b_2+b_3$. However, we also point out from detailed numerical analysis that when the less general condition $1.5|b_2|>a>|b_2|$ is satisfied, approximate synchronization (anticipatory, complete, lag) occurs.

III. ANTICIPATORY SYNCHRONIZATION FOR $\tau_2<\tau_1$

To start with, we first consider the transition to anticipatory synchronization in the coupled system (3). We have fixed the value of the feedback time delay τ_1 at $\tau_1=25.0$ while the other parameters are fixed as $a=0.16$, $b_1=0.2$, $b_2=0.1$, $b_3=0.1$ and the time delay in the coupling τ_2 is treated as the control parameter. With the above mentioned stability condition (13) and with the coupling delay τ_2 being less than the feedback delay τ_1, one can observe the transition to anticipatory synchronization. The time trajectory plot is shown in Fig. 4(a) depicting anticipatory synchronization, for the specific value of $\tau_2=20.0$ with the anticipating time equal to the difference between feedback and coupling delays, that is, $\tau=\tau_2-\tau_1$. The time-shifted plot Fig. 4(b), $x_2(t-\tau)$ vs $x_1(t)$, shows a concentrated diagonal line confirming the existence of anticipatory synchronization (we may note here that in all our numerical studies in this paper we leave out a sufficiently large number of transients, before presenting our figures).

Some time ago, Rosenbulm *et al.* [15] have introduced the notion of the similarity function $S_l(\tau)$ for characterizing the lag synchronization as a time-averaged difference between the variables x_1 and x_2 (with mean values being subtracted) taken with the time shift τ,

$$S_l^2(\tau)=\frac{\langle[x_2(t+\tau)-x_1(t)]^2\rangle}{[\langle x_1^2(t)\rangle\langle x_2^2(t)\rangle]^{1/2}}, \tag{14}$$

where $\langle x\rangle$ means the time average over the variable x. If the signals $x_1(t)$ and $x_2(t)$ are independent, the difference between them is of the same order as the signals themselves. If $x_1(t)=x_2(t)$, as in the case of complete synchronization, the similarity function reaches a minimum $S(\tau)=0$ for $\tau=0$. But for the case of a nonzero value of time shift τ, if $S_l(\tau)=0$, then there exists a time shift τ between the two signals $x_1(t)$ and $x_2(t)$ such that $x_2(t+\tau)=x_1(t)$, demonstrating lag synchronization.

In the present study, we have used the same similarity function $S_l(\tau)$ to characterize anticipatory synchronization with a negative time shift $-\tau$ instead of the positive time shift τ in Eq. (14). In other words, one may define the similarity function for anticipatory synchronization as

$$S_a^2(\tau)=\frac{\langle[x_2(t-\tau)-x_1(t)]^2\rangle}{[\langle x_1^2(t)\rangle\langle x_2^2(t)\rangle]^{1/2}}. \tag{15}$$

Then a minimum of $S_a(\tau)$, that is, $S_a(\tau)=0$, indicates that there exists a time shift $-\tau$ between the two signals $x_1(t)$ and $x_2(t)$ such that $x_2(t-\tau)=x_1(t)$, demonstrating anticipatory synchronization. Figure 5 shows the similarity function $S_a(\tau)$

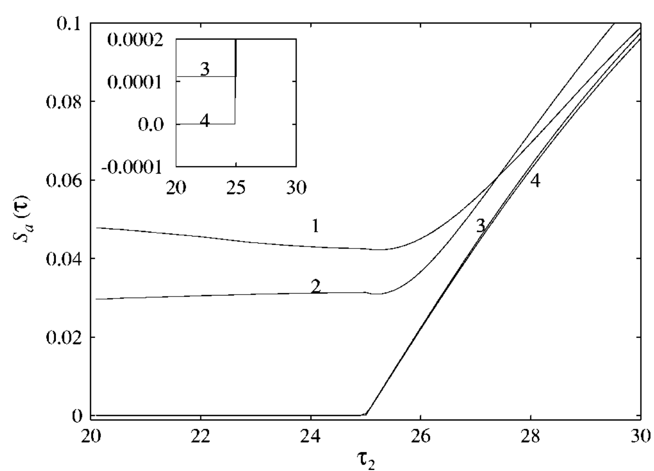

FIG. 5. Similarity function $S_a(\tau)$ for different values of b_2; the other system parameters are $a=0.16$, $b_1=0.2$, and $\tau_1=25.0$. (Curve 1, $b_2=0.18$, $b_3=0.02$; curve 2, $b_2=0.16$, $b_3=0.04$; curve 3, $b_2=0.15$, $b_3=0.05$; curve 4, $b_2=0.1$, $b_3=0.1$.)

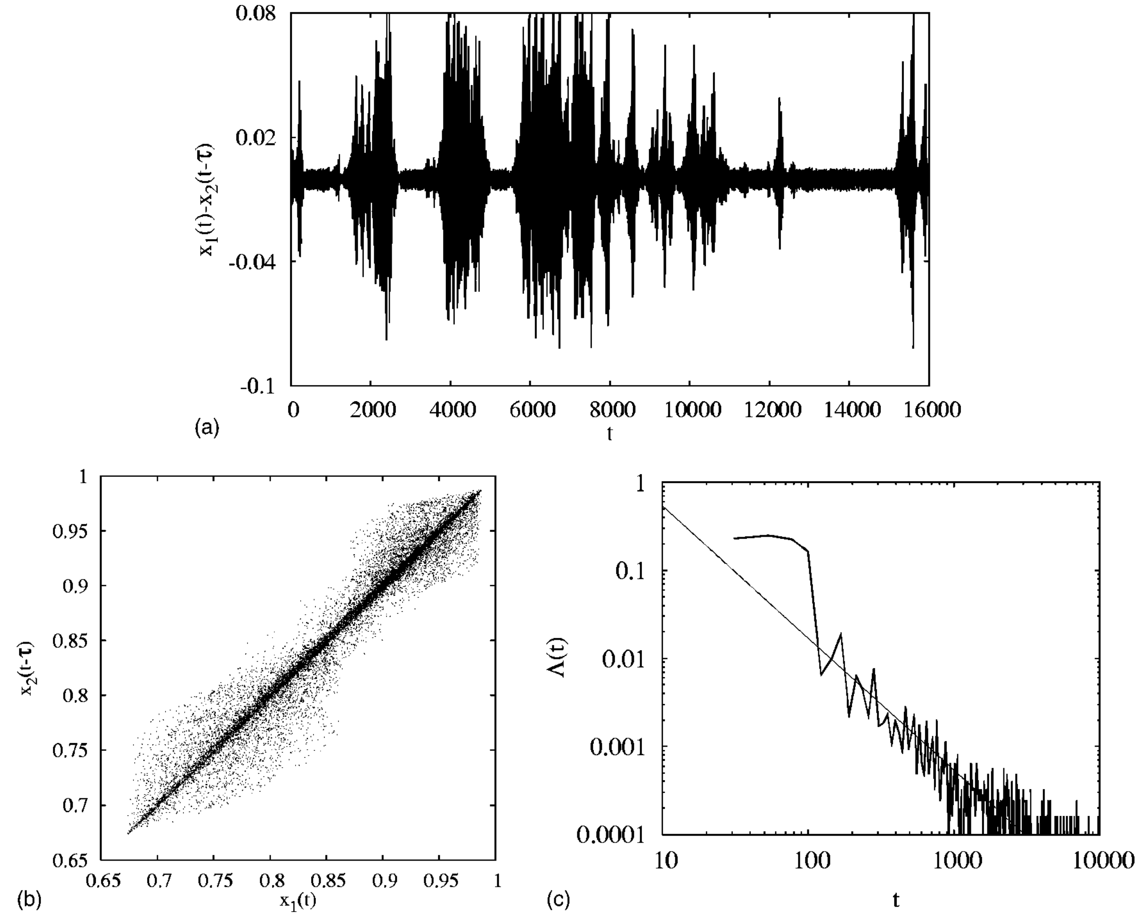

FIG. 6. (a) The time series $x_1(t)-x_2(t-\tau)$ for $b_2=0.17$ and $b_3=0.03$ with all other parameters as in Fig. 4 (so that the stability condition is violated for anticipatory synchronization). (b) Projection of $x_1(t)$ vs $x_2(t-\tau)$. (c) The statistical distribution of the laminar phase satisfying $-3/2$ power law scaling.

as a function of the coupling delay τ_2 for four different values of b_2, the parameter whose value determines the stability condition given by Eq. (13), while satisfying the parametric condition $b_1=b_2+b_3$. Curves 1 and 2 are plotted for the values of $b_2=0.18$ ($>a=0.16>a/1.5$) and $b_2=0.16$ ($=a>a/1.5$), respectively, where the minimum values of $S_a(\tau)$ are found to be greater than zero, indicating that there is no exact time shift between the two signals $x_1(t)$ and $x_2(t)$. Note that in the both cases the stringent stability condition (13) and the less stringent condition $a>|b_2|$ are violated. Curve 3 corresponds to the value of $b_2=0.15$ (which is less than a but greater than $a/1.5$), where the minimum value of $S_a(\tau)$ is almost zero, but not exactly zero (as may be seen in the inset of Fig. 5), indicating an approximate anticipatory synchronization $x_1(t)\approx x_2(t-\tau)$. On the other hand the curve 4 is plotted for the value of $b_2=0.1$ ($<a/1.5$), satisfying the general stability criterion Eq. (13). It shows that the minimum of $S_a(\tau)=0$, thereby indicating that there exists an exact time shift between the two signals demonstrating anticipatory synchronization. The anticipating time is found to be equal to the difference between the coupling and feedback delay times, that is, $\tau=\tau_2-\tau_1$. Note that $S_a(\tau)=0$ for all values of $\tau_2<\tau_1$, indicating anticipatory synchronization for a range of delay coupling. A further significance is that the anticipating time $\tau=|\tau_2-\tau_1|$ is an adjustable quantity as long as $\tau_2<\tau_1$, which can be tuned suitably to satisfy experimental situations.

Next, we show that the emergence of exact anticipatory synchronization is preceded by a region of approximate anticipatory synchronization, which is associated with the transition from on-off intermittency to a periodic structure in the laminar phase distribution [19] as a function of the parameter b_2. First we choose the value of b_2 as $b_2=0.17$ (with $b_1=0.2$ and $b_3=0.03$), above the value of $a=0.16$, such that the general stability criterion Eq. (13) as well as the less stringent condition $a>|b_2|$ are violated. Figure 6(a) shows the difference of $x_1(t)-x_2(t-\tau)$ vs t, exhibiting typical features of on-off intermittency [42,43] with the "off" state near the laminar phase and the "on" state showing a random burst. In Fig. 6(b) $x_1(t)$ is plotted against $x_2(t-\tau)$, where the distribution is scattered around the diagonal. To analyze the statistical features associated with the irregular motion, we calculated the distribution of laminar phases $\Lambda(t)$ with amplitude less than a threshold value $\Delta=0.005$ as was done in the statistical analysis of intermittency [42,43], where the power law behavior of the mean laminar length is calculated as a function of the control parameter. A universal asymptotic $-3/2$ power law distribution is observed in Fig. 6(c), which is quite typical for on-off intermittency.

Now, we choose the value of $b_2=0.15$, below the value of $a=0.16$ so that the less stringent condition $a>|b_2|$ is satisfied

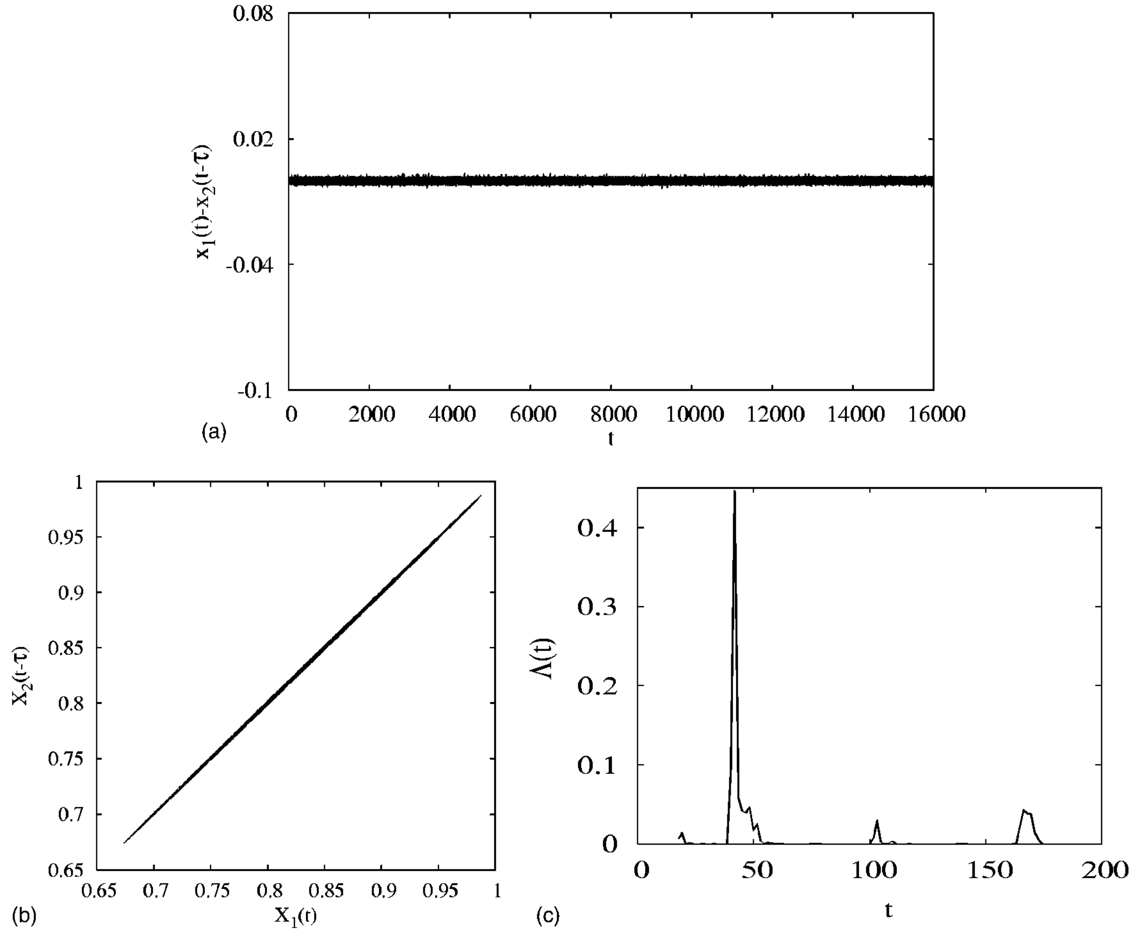

FIG. 7. (a) The time series $x_1(t)-x_2(t-\tau)$ for $b_2=0.15$ and $b_3=0.05$ with all other parameters fixed as in Fig. 4 [so that the less stringent condition $a>|b_2|$ is satisfied while Eq. (13) is violated]. (b) Projection of $x_1(t)$ vs $x_2(t-\tau)$. (c) The statistical distribution of the laminar phase showing a periodic structure.

while the general stability criterion Eq. (13) is violated and we carry out the same analysis as above. In Fig. 7(a), the difference of $x_1(t)-x_2(t-\tau)$ is plotted against time t, which is more regular and is much smaller in amplitude but not exactly zero, thereby implying an approximate anticipatory synchronization $x_1(t)\approx x_1(t-\tau)$. Figure 7(b) shows the plot of $x_1(t)$ vs $x_2(t-\tau)$, where the distribution is localized entirely on the diagonal, but not sharply on it. Earlier we noted that for this case the minimum of the similarity function $S_a(\tau)$ (curve 3, inset of Fig. 5) is nearly zero, but not exactly zero. The distribution of the laminar phase $\Lambda(t)$ is plotted in Fig. 7(c) as in Fig. 6(c). It shows a periodic structure in the distribution of the laminar phase, where the peaks occur approximately at $t=nT$, $n=1,2,\ldots$, where T is of the order of the period of the lowest periodic orbit of the uncoupled system (1). It should be remembered that the periodic behavior is associated with the statistical analysis, while the signals remain chaotic. Finally for the case $b_2=0.1$ ($<a/1.5$), which satisfies the stringent stability criterion (13), and where the similarity function vanishes exactly (curve 4 in Fig. 5), exact anticipatory synchronization occurs as confirmed in Fig. 4. Thus we find that the transition to exact anticipatory synchronization precedes a region of approximate anticipatory synchronization from the desynchronized state as the parameter b_2 changes. We have also demonstrated that the emergence of this approximate anticipatory synchronization from the desynchronized state is characterized by the transition of on-off intermittency to a periodic structure in the laminar phase distribution.

IV. COMPLETE SYNCHRONIZATION FOR $\tau_2=\tau_1$

Complete synchronization follows anticipatory synchronization as the value of the coupling time delay τ_2 becomes equal to the feedback time delay τ_1, when τ_2 is increased from a lower value. With $\tau_2=\tau_1$, the same stability criterion Eq. (13) holds good for this case of complete synchronization as well, with the same condition $b_1=b_2+b_3$. Figure 8(a) shows the time trajectory plot of $x_1(t)$ and $x_2(t)$, exhibiting synchronized evolution between them, which is also confirmed by the entirely localized diagonal line of $x_1(t)$ vs $x_2(t)$ as shown in Fig. 8(b). As in the case of anticipatory synchronization, we have found that the transition to complete synchronization precedes a region of approximate complete synchronization $[x_1(t)\approx x_2(t)]$ from the desynchronized state as the parameter b_2 varies. Here also we have identified that the emergence of approximate complete synchronization for the case $\tau_2=\tau_1$ is associated with a transition from on-off intermittency to a periodic structure in the laminar phase distribution as a function of the parameter b_2. In the next section

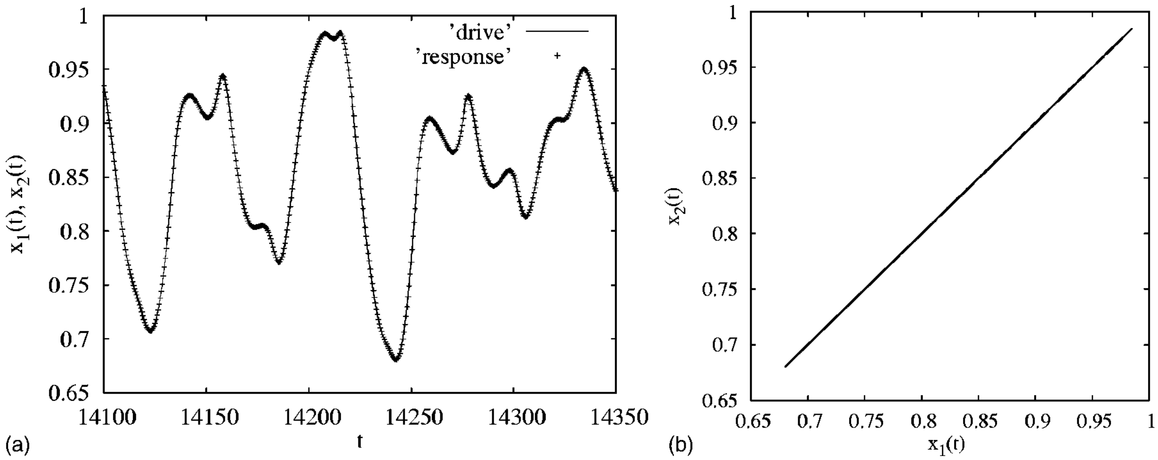

FIG. 8. Exact complete synchronization for the parameter values $a=0.16$, $b_1=0.2$, $b_2=0.1$, $b_3=0.1$, $\tau_1=25.0$, and $\tau_2=25.0$. Here the general stability criterion (13) is satisfied. (a) Time series plot of $x_1(t)$ and $x_2(t)$ and (b) synchronization manifold between $x_1(t)$ and $x_2(t)$. The response $x_2(t)$ follows identically the drive $x_1(t)$ without any time shift.

we will discuss the existence of lag synchronization for values of τ_2 greater than τ_1.

V. LAG SYNCHRONIZATION FOR $\tau_2 > \tau_1$

For coupling delay τ_2 greater than feedback delay τ_1, we find that the system (3) exhibits exact lag synchronization provided one satisfies the stringent stability criterion (13), with the lag time equal to the difference between the coupling and feedback delay times. Figure 9(a) shows the plot of $x_1(t)$ and $x_2(t)$ vs time t, where the response system lags the state of the drive system with constant lag time $\tau = |\tau_2 - \tau_1|$. Figure 9(b) shows the time-shifted plot of $x_1(t)$ and $x_2(t+\tau)$. However, in the region of less stringent stability condition, $1.5|b_2| < a < |b_2|$, approximate lag synchronization occurs as in the cases of anticipatory and complete synchronizations.

We have also calculated the similarity function $S_l(\tau)$ from Eq. (14) to characterize the lag synchronization. Figure 10 shows the similarity function $S_l(\tau)$ vs coupling delay τ_2 for four different values of b_2. Curves 1 and 2 show the similarity function $S_l(\tau)$ for the values of $b_2=0.18$ and 0.16, respectively. The minimum of the similarity function $S_l(\tau)$ occurs for values of $S_l(\tau) > 0$ and hence there is a lack of exact lag time between the drive and response signals indicating asynchronization. Curve 3 corresponds to the value of $b_2=0.15$ (which is less than a but greater than $a/1.5$), where the minimum values of $S_l(\tau)$ is almost zero, but not exactly zero (as may be seen in the inset of Fig. 10), so that $x_1(t) \approx x_2(t+\tau)$. However, for the value of $b_3=0.1$, for which the general condition (13) is satisfied, the minimum of the similarity function becomes exactly zero (curve 4) indicating that there is an exact time shift (Fig. 9) between drive and response signals $x_1(t)$ and $x_2(t)$, respectively, confirming the occurrence of lag synchronization.

We have also confirmed that as in the case of anticipatory synchronization, when the parameter b_2 varies, the onset of exact lag synchronization is preceded by a region of approximate lag synchronization, which is characterized by a transition from on-off intermittency of the desynchronized state to a periodic structure in the laminar phase distribution. For the value of $b_2=0.17$ [which violates the stability condition (13)

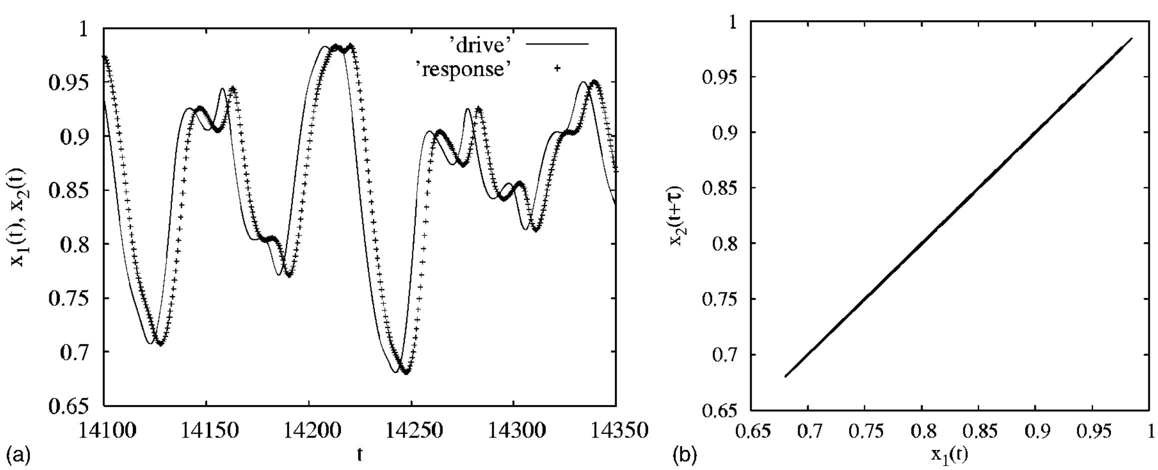

FIG. 9. Exact lag synchronization for the parameter values $a=0.16$, $b_1=0.2$, $b_2=0.1$, $b_3=0.1$, $\tau_1=25.0$, and $\tau_2=30.0$. Here the general stability criterion (13) is satisfied. (a) Time series plot of $x_1(t)$ and $x_2(t)$, and (b) synchronization manifold between $x_1(t)$ and $x_2(t+\tau)$. The response $x_2(t)$ lags the drive $x_1(t)$ with a time shift of $\tau=5.0$.

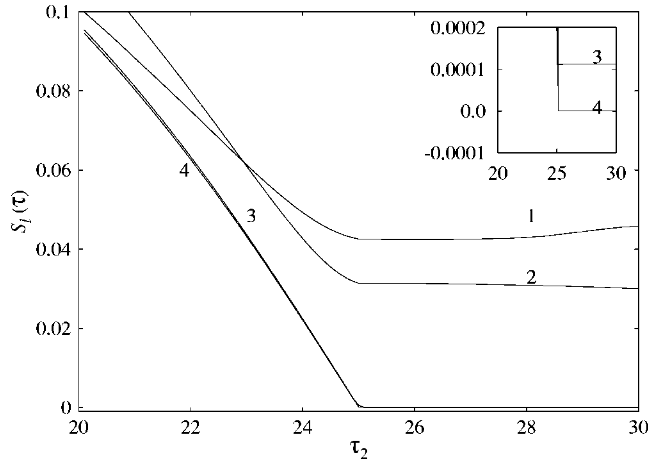

FIG. 10. Similarity function $S_l(\tau)$ for different values of b_2; the other system parameters are $a=0.16$, $b_1=0.2$, and $\tau_1=25.0$. (curve 1, $b_2=0.18$, $b_3=0.02$; curve 2, $b_2=0.16$, $b_3=0.04$; curve 3, $b_2=0.15$, $b_3=0.05$; and curve 4, $b_2=0.1$, $b_3=0.1$.)

as well as the less stringent condition $a>|b_2|$], Fig. 11(a) shows the difference of $x_1(t)-x_2(t+\tau)$ vs time t, exhibiting a typical on-off intermittency. In Fig. 11(b), $x_1(t)$ is plotted against $x_2(t+\tau)$, where the distribution is not concentrated along the diagonal. In Fig. 11(a), the laminar phase distribution $\Lambda(t)$ is characterized by an exponential $-3/2$ power law behavior as shown in Fig. 11(c). In order to show that there is a transition from on-off intermittency to periodic behavior in the laminar phase distribution corresponding to approximate lag synchronization, we have changed the value of b_2 from 0.17 to 0.15 [so that the less stringent condition $a>|b_2|$ is satisfied but not the general condition (13)], and examined the nature of laminar phase distribution $\Lambda(t)$. The difference between $x_1(t)$ and $x_2(t+\tau)$ is shown as a function of time t in Fig. 12(a), where there is only a laminar phase present for the threshold value $\Delta=0.002$ without any intermittent burst. The corresponding laminar phase distribution $\Lambda(t)$ is again characterized by a periodic structure as shown in Fig. 12(c). As in the case of approximate anticipatory synchronization, here also the peaks occur approximately at $t=nT$, $n=1,2,\ldots$, where T is roughly of the order of the period of the lowest periodic orbit of the uncoupled system (1). The time-shifted plot $x_1(t)$ vs $x_2(t+\tau)$ is shown in Fig. 12(b), where the distribution is concentrated along but not exactly on the diagonal line confirming the onset of approximate lag synchronization. As noted previously, for this case the minimum of the similarity function $S_l(\tau)$ is nearly zero but not exactly zero (curve 3, inset of Fig. 10). Finally for $b_2=0.1$, which satisfies the general stability criterion (13), we have exact lag synchronization as demonstrated in Figs. 9 and 10. Thus we find that as the parameter b_2 varies the transition to exact lag synchronization is preceded by a re-

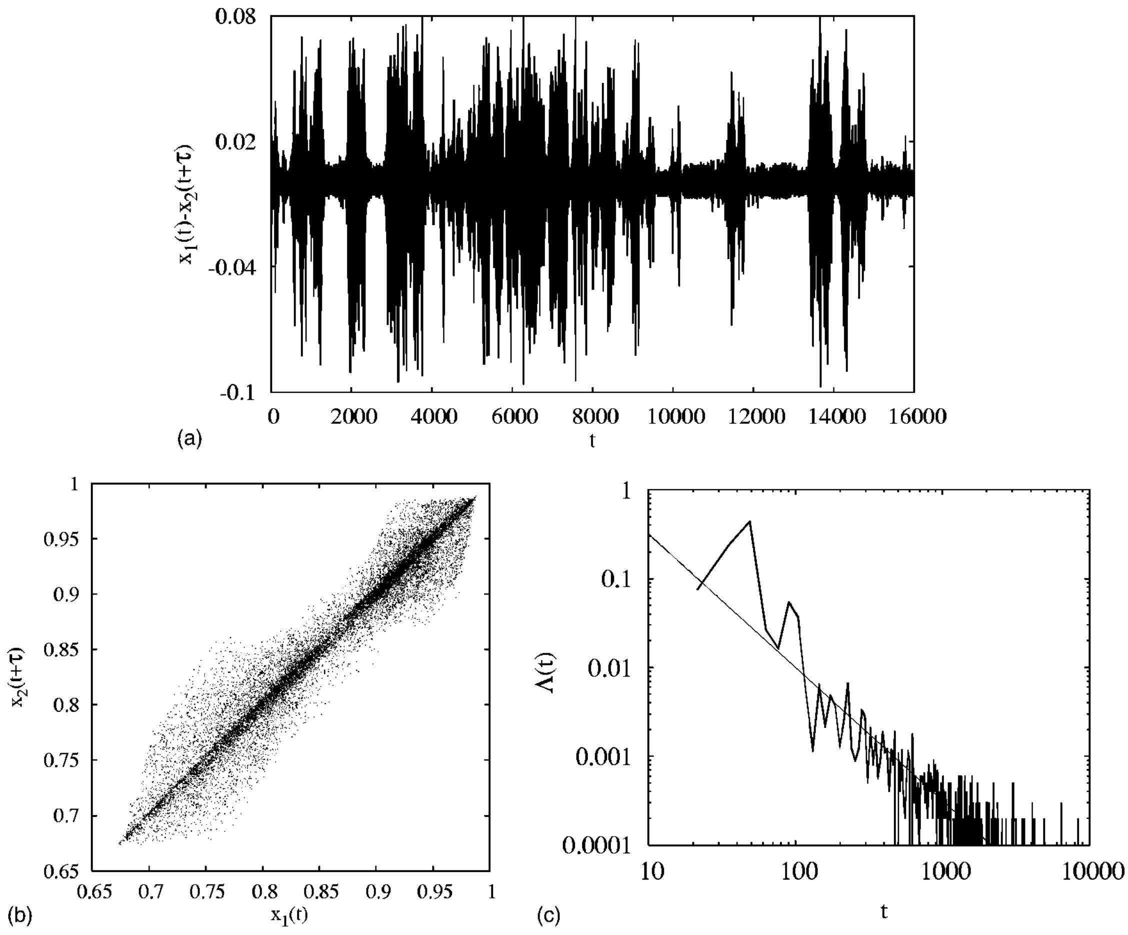

FIG. 11. (a) The time series $x_1(t)-x_2(t+\tau)$ for $b_2=0.17$ and $b_3=0.03$ with all other parameters as in Fig. 9 (so that the stability condition is violated). (b) Projection of $x_1(t)$ vs $x_2(t+\tau)$. (c) The statistical distribution of laminar phase satisfying $-3/2$ power law scaling.

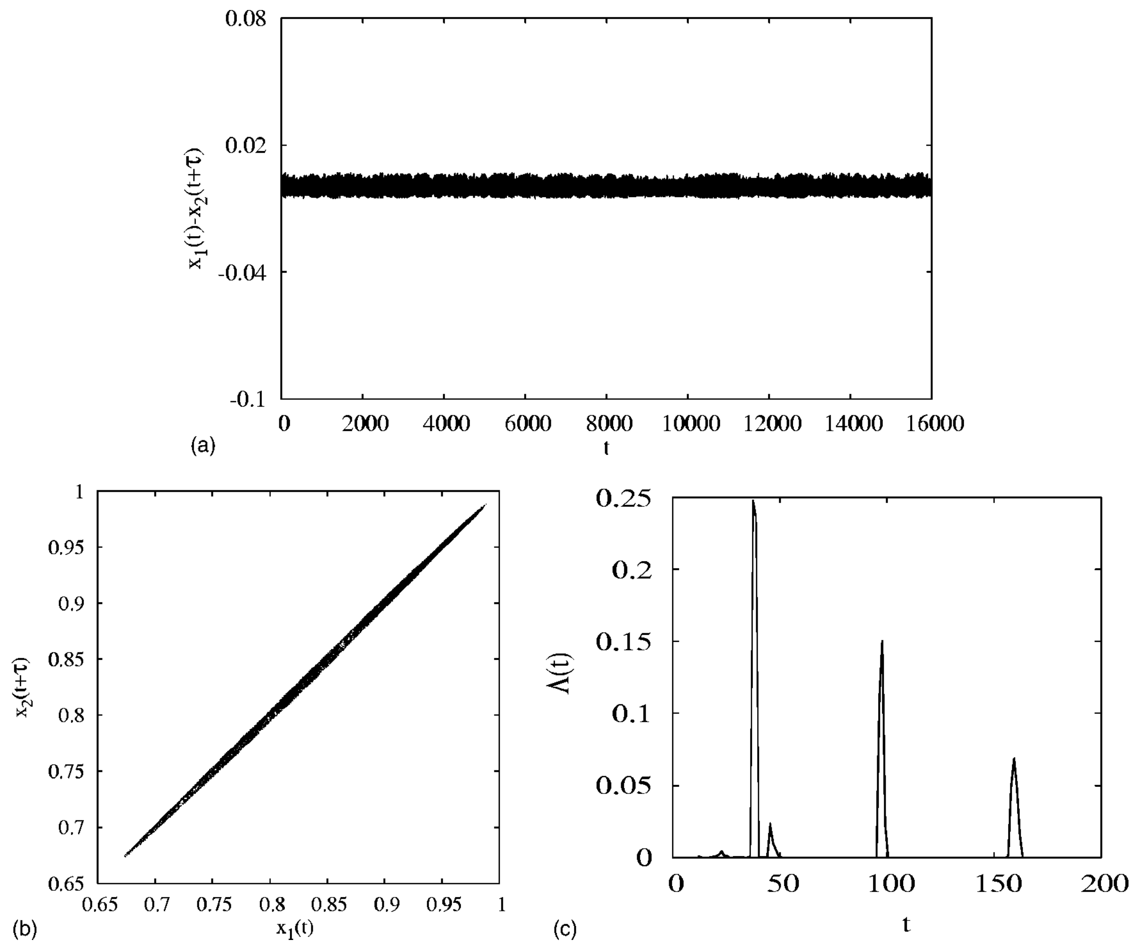

FIG. 12. (a) The time series $x_1(t)-x_2(t+\tau)$ for $b_2=0.15$ and $b_3=0.05$ so that the less stringent condition $a>|b_2|$ is satisfied while (13) is violated. (b) Projection of $x_1(t)$ vs $x_2(t+\tau)$. (c) The statistical distribution of the laminar phase showing periodic structure.

gion of approximate lag synchronization from the desynchronized state, where the latter is characterized by the transition from on-off intermittency to periodic structure in the laminar phase distribution.

VI. SUMMARY AND CONCLUSION

In this paper, we have shown the existence of transition from anticipatory synchronization to lag synchronization through complete synchronization in a single system of two coupled time-delay piecewise-linear oscillators with suitable stability condition and with the second time delay τ_2 in the coupling as the only control parameter with all the other parameters being kept fixed. We have also plotted corresponding similarity functions to characterize both the anticipatory and lag synchronization as well as complete synchronization. Further, when the parameter b_2 varies, we find that the transition to exact anticipatory, complete, or lag synchronization is preceded by a region of approximate anticipatory, complete, or lag synchronization from the desynchronized state, where the region of approximate synchronization is characterized by the transition from on-off intermittency to periodic structure in the laminar phase distribution.

Further, we have observed that in the region where the stringent stability condition (13) is satisfied, the minimum of the similarity function $S_a(\tau)$ attains the value zero for all values of $\tau_2<\tau_1$, indicating that exact anticipatory synchronization exists for a range of coupling delays τ_2 below τ_1. However, for approximate anticipatory synchronization (in the region $1.5|b_2|>a>|b_2|$) the minimum of the similarity function takes the value $S_a(\tau)\approx 0$, but not exactly zero, for $\tau_2<\tau_1$. Similarly, lag synchronization also occurs for a range of delay couplings τ_2 above τ_1. Another interesting aspect is that both the anticipating and lag time can be tuned to any desired value by changing the value of the coupling delay τ_2. Consequently, coupled time-delay systems of the type discussed in this paper have considerable physical relevance, particularly for secure communication purposes. We are now investigating the existence of similar phenomena in other piecewise-linear time-delay systems, including the time-delay Chua and Murali-Lakshman-Chua electronic circuits, the results of which will be published elsewhere.

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